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Aerodynamic sound in a relaxing medium

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In this paper we formulate the aerodynamic sound problem for a relaxing medium in a rather general way, independent of the details of the relaxation process. The medium is characterized by an appropriate relaxation time τ_0 and by a frozen (a_{f0}) and an equilibrium (a_{e0}) sound speed. The equation describing aerodynamic sound in such a medium is the familiar one describing acoustic waves in a non-equilibrium medium but subjected to aerodynamic sound sources expressed in terms of a frozen and an equilibrium form of the Lighthill stress tensor. The far-field result for both compact and non-compact sources in the frequency range $\omega \gg \tau_0^{-1}$ can be expressed as the ratio of far-field densities for the relaxing and non-relaxing propagation medium:

$$\frac{\rho}{\rho_L} = \exp\left[-\left(1-\frac{a_{e0}^2}{a_{f0}^2}\right)\frac{x}{2a_{f0}\tau_0}\right],$$

where x is the observation distance and the subscript L stands for 'Lighthill'. The result for the main radiated aerodynamic sound, which comes from sources in the range $\omega \ll \tau_0^{-1}$, essentially propagates in a manner described by the lower-order equilibrium waves, the diffusive effects from the higher-order waves being small, and the result for compact sources is a restatement of Lighthill's result in terms of the equilibrium propagation speed with the source region identically in equilibrium. For non-compact sources the propagation is still given by a_{e0} but the source region is now understood to encompass relaxation effects, the details of which are left unspecified.

1. Introduction

Aerodynamic sound generation (Lighthill 1952, 1962) in the presence of fluid inhomogeneities has been primarily discussed in terms of the effect of such inhomogeneities on the *aerodynamic sound source* (Crighton & Ffowcs Williams 1969; Ffowcs Williams 1969*a*, *b*; Strahle 1971; Hassan 1974; Crighton 1975). Such inhomogeneities in the source region, in general, enhance the sound generated. On the other hand, the effect of inhomogeneities, in the form of adjustment or relaxation processes, on the *propagation region* of an acoustic or Mach wave has received rather thorough discussion in the literature (Stokes 1851; Chu 1957; Moore & Gibson 1960; Vincenti & Kruger 1965; Marble 1970), although not in the context of aerodynamic sound generation. More recently Marble & Candel (1974) and Marble (1975) discussed the acoustic attenuation in fans and ducts by the vaporization of liquid droplets and found rather interesting possibilities for noise reduction. Inhomogeneities in the propagation region in general have a beneficial effect on the reduction of the emitted sound.

Our interest here is that the inhomogeneities, characterized by a kind of 'relaxation process', are present in both the source and the propagation region. Their presence in the propagation region may have been artificially induced (e.g. Marble 1975) in order to alter the propagation features of the sound that is of aerodynamic origin. The propagation region may also have been set up by chemical reactions that are very nearly in equilibrium (e.g. Vincenti & Kruger 1965), the acoustic waves emitted from a localized aerodynamic sound source displacing the propagation region out of equilibrium. For the purposes of understanding the general features of these problems, which may lead to aerodynamic noise reduction possibilities, it is thus appropriate to unify the various discussions in the literature and reconstruct Lighthill's (1952, 1962) aerodynamic sound theory for a relaxing medium.

One could, at the outset, specify the details of the particular relaxation process and then proceed to sort out theoretically the relaxation wave propagation operator and to interpret the deviation from such propagation effects as the aerodynamic sound sources. However, with the specification of details at the outset the source terms so obtained become rather unmanageably awkward. In the spirit of Lighthill's (1952, 1962) work, which encompasses both generality and simplicity, the details of the relaxation process need not be specified at the outset. We need to say only that the relaxation process is characterized by a relaxation time τ_0 and by an equilibrium and a frozen sound speed, a_{e0} and a_{f0}, respectively, in a propagation region which is otherwise in equilibrium.

We shall give a simple derivation of the aerodynamic sound theory in a relaxing medium in §2. The propagation is described in terms of a D'Alembertian characterized by the frozen sound speed relaxing towards one characterized by the equilibrium sound speed, while the appropriate source is interpreted in terms of a frozen Lighthill stress tensor $T_{ij,f}$, relaxing towards the equilibrium one $T_{ij,e}$. The appropriate Green's function for the three-dimensional relaxing wave propagation operator, which is known (Clarke 1964), is used to construct an exact integral for the aerodynamic sound in §3. The sound generated far from the source is then estimated in terms of the aerodynamic sound source.

2. Derivation of the aerodynamic sound equation for a relaxing medium

In what follows, we shall formulate the problem in terms of a single relaxation process characterized by a single relaxation time τ_0. This has certain merits in addition to simplicity as it is applicable to actual situations where the acoustic attenuation in an appropriate frequency range is due to a single physical process although the medium itself may possess multiple excitable relaxations or reactions. For instance, the vaporization process is found to be the dominant attenuation mechanism in a multiphase medium (Marble 1975).

We begin by writing down the continuity equation for the fluid

$$\partial\rho/\partial t + \partial(\rho u_i)/\partial x_i = Q, \tag{2.1}$$

where ρ is the density, u_i the velocity vector, Q the local mass production rate and x_i and t are the co-ordinates and time, respectively. The momentum equations for the fluid are

$$\frac{\partial}{\partial t}\rho u_i + \frac{\partial}{\partial x_j}\rho u_i u_j = -\frac{\partial}{\partial x_j}p_{ij} + F_i, \tag{2.2}$$

where p_{ij} is the stress tensor and F_i the volumetric force acting on the fluid.

We now proceed to draw an analogy between the nonlinear problem above, (2.1) and (2.2), and the linear theory of acoustics in a relaxing or reacting medium. The governing equation for the linear problem, which was first obtained by Stokes (1851), is (see, for instance, Vincenti & Kruger 1965)

$$\tau_0 \partial(\Box_f^2 \rho)/\partial t + \Box_e^2 \rho = 0, \tag{2.3}$$

where the frozen wave operator is defined as

$$\Box_f^2 \equiv \partial^2/\partial t^2 - a_{f0}^2 \nabla^2,$$

a_{f0} denoting the ambient frozen sound speed and ∇^2 the Laplacian $\partial^2/\partial x_i \partial x_i$, while the equilibrium wave operator is

$$\Box_e^2 \equiv \partial^2/\partial t^2 - a_{e0}^2 \nabla^2,$$

a_{e0} denoting the ambient equilibrium sound speed. To recast (2.1) and (2.2) into a form where the left-hand sides exhibit the same relaxation wave operator as in (2.3), we define the following Lighthill stress tensors:

$$T_{ij,f} \equiv \rho u_i u_j + p_{ij} - a_{f0}^2 \rho \delta_{ij} \quad \text{(frozen)},$$

$$T_{ij,e} \equiv \rho u_i u_j + p_{ij} - a_{e0}^2 \rho \delta_{ij} \quad \text{(equilibrium)}.$$

Thus (2.2) can be written exactly as

$$\frac{\partial(\rho u_i)}{\partial t} + a_{f0}^2 \frac{\partial \rho}{\partial x_i} = -\frac{\partial T_{ij,f}}{\partial x_j} + F_i \tag{2.4}$$

and

$$\frac{\partial(\rho u_i)}{\partial t} + a_{e0}^2 \frac{\partial \rho}{\partial x_i} = -\frac{\partial T_{ij,e}}{\partial x_j} + F_i. \tag{2.5}$$

By eliminating ρu_i between (2.1) and (2.4) we obtain an equation appropriate for the frozen acoustic problem while eliminating ρu_i between (2.1) and (2.5) gives an equation appropriate for the equilibrium acoustic problem. Combining these in a manner guided by the linear problem (2.3), we obtain

$$\tau_0 \frac{\partial}{\partial t} \Box_f^2 \rho + \Box_e^2 \rho = \left(\tau_0 \frac{\partial}{\partial t} + 1\right) \left(\underbrace{\frac{\partial Q}{\partial t}}_{\text{'mono-pole'}} - \underbrace{\frac{\partial F_i}{\partial x_i}}_{\text{'di-pole'}}\right) + \frac{\partial^2}{\partial x_i \partial x_j} \underbrace{\left(\tau_0 \frac{\partial T_{ij,f}}{\partial t} + T_{ij,e}\right)}_{\text{'quadrupole'}}. \tag{2.6}$$

This is the equation describing a propagation medium which is in equilibrium except where small amplitude acoustic waves displace the medium out of equilibrium as described by the left side of (2.6). Such acoustic waves are generated by the nonlinear unsteady motions, which are contained in a finite region. More precisely, such relaxation–acoustic waves are generated by the 'external' fluctuating sources of mass Q, forces F_i and stresses $T_{ij,f}$ and $T_{ij,e}$ on the right side of (2.6). The propagation medium is characterized by the relaxation time τ_0 and the frozen and equilibrium sound speeds a_{f0} and a_{e0}, respectively (where $a_{f0} > a_{e0}$). We have obtained (2.6) entirely independently of the details of the relaxation process. If the sources on the right of (2.6) are known, the problem of aerodynamic sound generation in a relaxing medium may be solved in a manner similar to acoustical problems in a relaxing or reacting gas (see, for instance, Clarke 1964).

3. The radiated aerodynamic sound field

It is known from acoustics in a relaxing medium (Chu 1957; Clarke 1964) that the Green's function for (2.6), like that for the piston problem associated with the homogeneous part of (2.6), admits a wave propagation field describable by its behaviour in the vicinity of the two spherical shells associated with the two sound speeds a_{f0} and a_{e0}. The leading characteristic is associated with the highest-order, frozen wave a_{f0}, the medium remaining undisturbed ahead of it and a discontinuous jump, just as in ordinary acoustics, occurring at the wave itself. However, in a relaxing medium the signal on the frozen wave front $R \sim a_{f0}t$ is damped according to

$$\frac{1}{R}\exp\left\{-\frac{1}{2}\left(1-\frac{a_{e0}^2}{a_{f0}^2}\right)\frac{R}{a_{f0}\tau_0}\right\}, \tag{3.1}$$

where R is the distance from the source. For large distances from the source, the bulk of the signal is carried along the equilibrium wave $R \sim a_{e0}t$, but the signals there behave in a diffusive manner according to

$$\frac{1}{R}\frac{1}{\{4[\frac{1}{2}\tau_0(a_{f0}^2-a_{e0}^2)]R/a_{e0}\}^{\frac{1}{2}}}\exp\left\{-\frac{(R-a_{e0}t)^2}{4[\frac{1}{2}\tau_0(a_{f0}^2-a_{e0}^2)]t}\right\}. \tag{3.2}$$

The geometric factor R^{-1} is augmented by the 'diffusional' decay $R^{-\frac{1}{2}}$. The signals along the equilibrium wave are diffused, reaching maxima along the 'front' $R \sim a_{e0}t$. It can be surmised intuitively (Lighthill & Whitham 1955; Whitham 1959) that the effect of the lower-order waves on the higher-order ones provides 'damping', while the effect of the higher-order waves on the lower-order ones is to provide a 'diffusional' behaviour.

In the absence of relaxation, the Green's function wave front decays simply as $(a_0^2R)^{-1}$, where a_0 is the sound speed in the propagation medium. It can be seen that an identically equilibrium propagation medium, where $a_0 = a_{e0}$, is singular in that the diffusional effect of the higher-order frozen waves is entirely absent. This is the case only when $\tau_0 \equiv 0$. For aerodynamic sound whose frequency content is much larger than τ_0^{-1}, the modified Green's function in the propagation region has the behaviour shown by (3.1), which is advantageous to aerodynamic sound reduction or 'absorption'. The question that naturally arises is that of the circumstances under which this situation could be exploited for practical use. At the outset, one could envision that the use of (3.1) would best be suited to 'shroud' localized regions of intense sound sources. Since the relation (3.1) requires that the frequency of the aerodynamic sound be large compared with τ_0^{-1}, a τ_0^{-1} corresponding approximately to 330 Hz is achievable, for instance, by the injection of water droplets of radius about 0·7 μm at a weight fraction of 1 % into air at about 25 °C at 1 atmosphere (see Marble 1975; Marble & Candel 1974). In this case $a_{f0}^2/a_{e0}^2 - 1 \cong 0{\cdot}21$ and for $a_{f0} \simeq 440$ m/s the e-folding distance for the Green's function is about 1 m. The relation (3.2) requires that the aerodynamic sound frequency content be small compared with τ_0^{-1}, however the diffusive effects are relatively small in this situation and the Green's function in this frequency range is just that for the identically equilibrium medium corresponding to a propagation speed of a_{e0}. The present study, which places 'absorption' and 'dispersion' in the context of aerodynamic sound, is directed towards understanding how aerodynamic sound in the undesirable frequency ranges could be absorbed with that in the less undesirable ranges essentially left unaltered and propagated into the far field but at a slightly lower speed.

Although we could formally write the exact expression representing ρ in (2.6) in terms of the Green's function and the sources on the right side of (2.6) (see, for instance, Clarke 1964), the result is rather cumbersome. It will be more illuminating to consider the problem approximately and elucidate the behaviour of the aerodynamic sound 'heard' on the frozen wave front and on the equilibrium wave, which ultimately carries the bulk of the signal. To this end, we employ the Fourier transform in time, for which much of the formalism discussed by Crighton (1975, §4) can be used. We then consider the problem and relevant approximations directly in $\mathbf{x}, \omega$ space, where $\mathbf{x}$ is the spatial position and ω is a particular frequency. For convenience, we use the notation introduced by Crighton (1975), in which the argument of the function denotes the status of its Fourier transform, rather than introducing new symbols for the functions themselves. The Fourier transform of $f(\mathbf{x}, t)$ with respect to t is defined as

$$f(\mathbf{x}, \omega) = \int f(\mathbf{x}, t) e^{i\omega t} dt,$$

and its inverse transform is

$$f(\mathbf{x}, t) = \frac{1}{2\pi} \int f(\mathbf{x}, \omega) e^{-i\omega t} d\omega.$$

We rewrite the equation (2.6) for aerodynamic sound in a relaxing medium as

$$\left[\tau_0 \left(\frac{a_{f0}}{a_{e0}}\right)^2 \frac{\partial}{\partial t} + 1\right] \nabla^2 \rho - \left[\tau_0 \frac{\partial}{\partial t} + 1\right] \frac{1}{a_{e0}^2} \frac{\partial^2 \rho}{\partial t^2} = A(\mathbf{x}, t), \tag{3.3}$$

where the source terms are defined as

$$A(\mathbf{x}, t) = \left\{ -\frac{1}{a_{e0}^2} \left(\tau_0 \frac{\partial}{\partial t} + 1\right) \left(\frac{\partial Q}{\partial t} - \frac{\partial F_i}{\partial x_i}\right) - \frac{1}{a_{e0}^2} \frac{\partial^2}{\partial x_i \partial x_j} \left(\tau_0 \frac{\partial T_{ij,f}}{\partial t} + T_{ij,e}\right)\right\}. \tag{3.4}$$

The special forms of frozen and equilibrium source terms are defined as

$$\begin{pmatrix} A_f(\mathbf{x}, t) \\ A_e(\mathbf{x}, t) \end{pmatrix} = -\begin{pmatrix} a_{f0}^{-2} \\ a_{e0}^{-2} \end{pmatrix} \left[\left(\frac{\partial Q}{\partial t} - \frac{\partial F_i}{\partial x_i}\right) + \frac{\partial^2}{\partial x_i \partial x_j} \begin{pmatrix} T_{ij,f} \\ T_{ij,e} \end{pmatrix}\right].$$

Applying the Fourier transform in time to (3.3) also gives an inhomogeneous Helmholtz equation

$$(\nabla^2 + \bar{k}_0^2) \rho(\mathbf{x}, \omega) = \mathscr{A}(\mathbf{x}, \omega) \tag{3.5}$$

but the effective wavenumber and source terms are now

$$\bar{k}_0^2 = \frac{\omega^2}{a_{e0}^2} \frac{1 - i\omega\tau_0}{1 - i\omega\tau_0 (a_{f0}/a_{e0})^2},$$

$$\mathscr{A}(\mathbf{x}, \omega) = \frac{A(\mathbf{x}, \omega)}{1 - i\omega\tau_0 (a_{f0}/a_{e0})^2}$$

and

$$A(\mathbf{x}, \omega) = -\frac{1}{a_{e0}^2} (1 - i\omega\tau_0) \left[-i\omega Q(\mathbf{x}, \omega) - \frac{\partial}{\partial x_i} F_i(\mathbf{x}, \omega)\right] - \frac{1}{a_{e0}^2} \frac{\partial^2}{\partial x_i \partial x_j} \times [T_{ij,e}(\mathbf{x}, \omega) - i\omega\tau_0 T_{ij,f}(\mathbf{x}, \omega)].$$

We note here that the special forms $\mathscr{A}_f(\mathbf{x},\omega)$ and $\mathscr{A}_e(\mathbf{x},\omega)$ are identical to $A_f(\mathbf{x},\omega)$ and $A_e(\mathbf{x},\omega)$, respectively:

$$\mathscr{A}_f(\mathbf{x},\omega) = A_f(\mathbf{x},\omega) = -\frac{1}{a_{f0}^2}\left[-i\omega Q(\mathbf{x},\omega) - \frac{\partial}{\partial x_i}F_i(\mathbf{x},\omega) + \frac{\partial^2}{\partial x_i \partial x_j}T_{ij,f}(\mathbf{x},\omega)\right], \tag{3.6}$$

$$\mathscr{A}_e(\mathbf{x},\omega) = A_e(\mathbf{x},\omega) = -\frac{1}{a_{e0}^2}\left[-i\omega Q(\mathbf{x},\omega) - \frac{\partial}{\partial x_i}F_i(\mathbf{x},\omega) + \frac{\partial^2}{\partial x_i \partial x_j}T_{ij,e}(\mathbf{x},\omega)\right]. \tag{3.7}$$

Equations (3.6) and (3.7) do not imply that the sources are themselves either 'frozen' or in 'equilibrium', but that they appear in the above form in their respective contributions to the frozen and equilibrium waves, as we shall soon see. The details of the relaxation process of the sources need not be specified as yet but are entirely encompassed in (3.6) and (3.7).

The Green's function for the three-dimensional Helmholtz equation (3.5) is

$$G(\mathbf{x},\omega) = -(4\pi x)^{-1}\exp(i\bar{k}_0 x), \tag{3.8}$$

where $x = |\mathbf{x}|$. Thus, using (3.8), we obtain

$$\rho(\mathbf{x},\omega) = -\frac{1}{4\pi}\int \mathscr{A}(\mathbf{y},\omega)\frac{\exp(i\bar{k}_0|\mathbf{x}-\mathbf{y}|)}{|\mathbf{x}-\mathbf{y}|}d\mathbf{y}, \tag{3.9}$$

where $\mathbf{y}$ denotes the source-point co-ordinates and $\mathbf{x}$ the observation-point co-ordinates. Following Crighton (1975), if we introduce y_r and $\mathbf{y}_s$, in the direction of and normal to $\mathbf{x}$, respectively, the approximation to order x^{-1} gives

$$\rho(\mathbf{x},\omega) = -\frac{\exp(i\bar{k}_0 x)}{4\pi x}\int \mathscr{A}(y_r,\mathbf{y}_s,\omega)\exp(-i\bar{k}_0 y_r)\,dy_r\,d\mathbf{y}_s. \tag{3.10}$$

The formal solution of (3.10) is then

$$\rho(\mathbf{x},t) = -\frac{1}{8\pi^2 x}\int \exp[i(\bar{k}_0 x - \omega t)]\int \mathscr{A}(y_r,\mathbf{y}_s,\omega)\exp(-i\bar{k}_0 y_r)\,dy_r\,d\mathbf{y}_s\,d\omega. \tag{3.11}$$

Direct interpretation of (3.11) is difficult and approximate considerations suffice. Our interest here is in the behaviour of the solution near the frozen wave fronts and in the features of the bulk of the radiated aerodynamic sound located near the equilibrium waves. The estimation of ρ near the frozen wave front is obtained by considering $\rho(\mathbf{x},\omega)$ for large $\omega\tau_0$. Thus, for large $\omega\tau_0$

$$\rho(\mathbf{x},t) \cong -\frac{1}{8\pi^2 x}\exp\left[-\left(1-\frac{a_{e0}^2}{a_{f0}^2}\right)\frac{x}{2a_{f0}\tau_0}\right]\int \exp[ik_{f0}(x-a_{f0}t)]$$
$$\times\int A_f(y_r,\mathbf{y}_s,\omega)\exp(-ik_{f0}y_r)\exp\left[\left(1-\frac{a_{e0}^2}{a_{f0}^2}\right)\frac{y_r}{2a_{f0}\tau_0}\right]dy_r\,d\mathbf{y}_s\,d\omega, \tag{3.12}$$

where the appropriate wavenumber here is $k_{f0} = \omega/a_{f0}$. It is obvious from (3.12) that the radiated aerodynamic sound with frequencies large compared with τ_0^{-1} is exponentially damped along the frozen wave fronts $x = a_{f0}t$ and that the effective sources have the frozen *form* of the Lighthill sources (but are not necessarily frozen). This conclusion is obtained without any assumption about the compactness of sources. However, in order to cast the source integral into more familiar forms it will be further simplified by considering the limiting cases of compact and non-compact sources.

For the compact case, $k_{f0} \to 0$, or more precisely, $l\omega/a_{f0} \to 0$, where l is a typical length scale of the source region. For the source frequencies large compared with τ_0^{-1} in our consideration of the behaviour of the sound radiated along the frozen wave fronts, this implies that $l/a_{f0}\tau_0 \to 0$. Thus the exponential factor in the source integral (3.12), i.e. $(1-a_{e0}^2/a_{f0}^2)y_r/2a_{f0}\tau_0$, is indeed small and the source integral becomes

$$\int \exp[i\omega(x/a_{f0}-t)] \int A_f(y_r, \mathbf{y}_s, \omega)\, dy_r\, d\mathbf{y}_s\, d\omega,$$

giving the interpretation that the frozen form of the Lighthill source integral is exponentially damped:

$$\rho(\mathbf{x},t) \cong -\frac{1}{8\pi^2 x} \exp\left[-\left(1-\frac{a_{e0}^2}{a_{f0}^2}\right)\frac{x}{2a_{f0}\tau_0}\right] \int A_{f,f}(\mathbf{y}, t - x/a_{f0})\, d\mathbf{y}. \tag{3.13}$$

Since for compact sources the source frequency corresponds to that of the radiated aerodynamic sound frequency, the situation $\omega\tau_0 \gg 1$ also refers to the source region. The relaxation process for the sources is thus in the frozen situation as well and $A_f = A_{f,f}$. The reduction of aerodynamic sound by absorption given by (3.13) can thus be cast into a very simple form for compact sources:

$$\frac{\rho}{\rho_L} = \exp\left[-\left(1-\frac{a_{e0}^2}{a_{f0}^2}\right)\frac{x}{2a_{f0}\tau_0}\right] \tag{3.14}$$

in the frequency range $\omega \gg \tau_0^{-1}$.

For non-compact sources $\omega \to 0$ such that $\omega l/u \to 0$, where u is a typical velocity scale of the sources. For the present $\omega \gg \tau_0^{-1}$ situation $l/u\tau_0 \to 0$ or $m_f^{-1}(l/a_{f0}\tau_0) \to 0$, where the Mach number $m_f = u/a_{f0} \gg 1$. Thus it is consistent to have $l/a_{f0}\tau_0 = O(1)$ at most. The exponential factor in the source integral (3.12) can still be rendered negligible by noting that $1-a_{e0}^2/a_{f0}^2 \ll 1$ in most practical situations. In this case, the source integral in (3.12) reduces to

$$a_{f0}\int \exp[ik_{f0}(x-a_{f0}t)] \int A_f(y_r, \mathbf{y}_s, 0) \exp[-ik_{f0}y_r]\, dy_r\, d\mathbf{y}_s\, dk_{f0},$$

which is interpreted as

$$-4\pi^2 a_{f0} \int A_f(y_r = x - a_{f0}t, \mathbf{y}_s, t)\, d\mathbf{y}_s\, dt.$$

Since $l/u\tau_0 \ll 1$, i.e. the relaxation length is large compared with the source length scale, again $A_f = A_{f,f}$, so that the statement (3.14) also holds for non-compact sources when the propagation medium is characterized by the relaxation time τ_0 for the sources in the spectral range $\omega \gg \tau_0^{-1}$.

We have shown that aerodynamic sound in the spectral range $\omega \gg \tau_0^{-1}$ propagates with the a_{f0} waves but is exponentially damped and becomes 'absorbed' in a distance of order $2(1-a_{e0}^2/a_{f0}^2)^{-1}a_{f0}\tau_0$ from the source. The question which naturally arises is how the bulk of the aerodynamic sound is propagated. Earlier studies of wave propagation in a relaxing medium (Chu 1957; Moore & Gibson 1960; Clarke 1964; Vincenti & Kruger 1965) involving wave hierarchies (Lighthill & Whitham 1955; Whitham 1959, 1974) have shown that the main disturbance is propagated along the equilibrium waves a_{e0}. For an estimate of the behaviour of $\rho(\mathbf{x},t)$ at large values of $x/a_{e0}\tau_0$ in the vicinity of $x = a_{e0}t$, expansions about $\omega\tau_0 = 0$ are made. The first approximation gives the aerodynamic sound problem for an equilibrium medium

$$\rho(\mathbf{x},t) \cong -\frac{1}{8\pi^2 x}\int \exp[ik_{e0}(x-a_{e0}t)] \int A_e(y_r, \mathbf{y}_s, \omega) \exp(-ik_{e0}y_r)\, dy_r\, d\mathbf{y}_s\, d\omega$$

(where $k_{e0} = \omega/a_{e0}$ is the appropriate wavenumber), from which the compact and non-compact source limits could be obtained. In order to obtain the effect of the higher-order frozen waves on the behaviour in the vicinity of the equilibrium waves expansion in $\omega\tau_0$ to the next order is required, giving

$$\rho(\mathbf{x}, t) \cong -\frac{1}{8\pi^2 x}\int \exp[ik_{e0}(x-a_{e0}t) - k_{e0}^2\mathscr{D}x/a_{e0}]\int A_e(y_r, \mathbf{y}_s, \omega)\exp[-ik_{e0}y_r + k_{e0}^2\mathscr{D}y_r/a_{e0}]\,dy_r\,d\mathbf{y}_s\,d\omega, \qquad (3.15)$$

where an effective 'diffusivity' is defined as $\mathscr{D} = \frac{1}{2}\tau_0(a_{f0}^2 - a_{e0}^2)$. Equation (3.15) then forms the basis for our studies of compact and non-compact sources. Again, for compact sources $k_{e0}l \to 0$ and $k_{e0}^2\mathscr{D}y_r \to 0$, so that (3.15) becomes, with the use of the convolution theorem,

$$\rho(\mathbf{x}, t) \cong -\frac{\mathscr{D}}{(4\mathscr{D}x/a_{e0})^{\frac{3}{2}}}\int \exp\left(-\frac{(t - x/a_{e0} - \tau)^2}{4(\mathscr{D}x/a_{e0})/a_{e0}^2}\right)\int A_{e,e}(\mathbf{y}, \tau)\,d\mathbf{y}\,d\tau. \qquad (3.16)$$

The main disturbances, corresponding to the sources in the spectral range $\omega \ll \tau_0^{-1}$, propagate along the a_{e0} waves and are diffused by the effects of the higher-order waves. Since $\omega\tau_0 \ll 1$ applies in the source region as well, i.e. $A_e = A_{e,e}$, the sources behave as if they are entirely in equilibrium. The compact source result (3.16) can be interpreted as follows:

$$(\mathscr{D}/a_{e0})\int A_e(\mathbf{y}, t)\,d\mathbf{y}$$

is a distribution of point sources located at $x/a_{e0} \to 0$. In the far field, the disturbances are propagated along the $t = x/a_{e0}$ waves and are diffused, the diffusive structure about each disturbance pulse being $(4\mathscr{D}x/a_{e0})^{\frac{1}{2}}$. However, the diffusive effects are small for the 'low' frequency waves and the main aerodynamic sound is given by

$$\rho(\mathbf{x}, t) = -\frac{1}{8\pi^2 x}\int A_{e,e}(\mathbf{y}, t - x/a_{e0})\,d\mathbf{y}. \qquad (3.17)$$

For non-compact sources, $\omega l/u \to 0$ such that $\omega l/(a_{e0}m_e) \to 0$, where $m_e = u/a_{e0} \gg 1$ and $\omega l/a_{e0} = k_{e0}l$ is arbitrary but at most $O(1)$. The exponential factor in the source integral of (3.15), $\frac{1}{2}(a_{f0}^2/a_{e0}^2 - 1)(\omega\tau_0)(\omega y_r/a_{e0})$, is negligible because $\omega\tau_0 \ll 1$. Thus

$$\rho(\mathbf{x}, t) \cong \frac{2\pi\mathscr{D}}{(4\pi\mathscr{D}x/a_{e0})^{\frac{3}{2}}}\int \exp\left(-\frac{(x - a_{e0}t - \xi)^2}{4\mathscr{D}x/a_{e0}}\right)\iint A_e(y_r = \xi, \mathbf{y}_s, t)\,d\mathbf{y}_s\,dt\,d\xi. \qquad (3.18)$$

Since both $\omega l/u \to 0$ and $\omega\tau_0 \to 0$, the ratio $l/u\tau_0$ is left arbitrary. Thus, depending on the magnitude of $l/u\tau_0$, the sources in A_e could conceivably be undergoing effects of relaxation. The effects of the diffusivity are again small, and the main aerodynamic sound radiates in the manner described by (3.17), except that the source description is A_e rather than $A_{e,e}$, which requires consideration of the details of the relaxation process in the non-compact source region.

Considering the form of the sound sources A_f and A_e further, we note that, in a pure gas in the absence of external sources of mass-flux fluctuation Q and forces F_i, A_f is due entirely to the stress tensor $T_{ij,f}$. Now, for a relaxing medium, for instance injected vaporizing droplets in a gas, the contributions to Q and F_i would be due to the mass exchange between the gas and droplets and the force acting on the gas due to droplet drag. At equilibrium, as for 'low' frequency compact sources, both of these vanish, so that the contribution to the source integral $A_{e,e}$ would come from

$T_{ij,e}$. However, for 'low' frequency non-compact sources, A_e necessarily encompasses relaxation effects in the source region. One could, of course, make the frequency 'high' or 'low' for the relaxing medium by regulating τ_0^{-1}.

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